L'Hospital Rule:

A special kind of rule that allows evaluating limits of indeterminate forms.

If $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\pm\infty$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

Basic Indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$

Other Indeterminate forms: $0 \cdot \infty$, 1^{∞}, 0^{0}, $\infty - \infty$, ∞^{0}.

Examples for Basic Indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$

It looks like fractions, but we don't use Quotient Rule

① Find $\lim_{x \to 1} \frac{\ln x}{2x-2}$.

Note: $\frac{\ln 1}{2-2} = \frac{0}{0}$ ⇝ indeterminate form.

So, we apply L'Hospital Rule:

$$\lim_{x \to 1} \frac{\ln x}{2x-2} = \lim_{x \to 1} \frac{1/x}{2} = \lim_{x \to 1} \frac{1}{2x} = \frac{1}{2.1} = \frac{1}{2}$$

② Find $\lim_{x \to 0} \frac{\sin 3x}{\tan 4x}$.

$$\lim_{x \to 0} \frac{\sin 3x}{\tan 4x} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{\cos 3x \cdot 3}{\sec^2 4x \cdot 4} = \frac{1 \cdot 3}{1 \cdot 4} = \frac{3}{4}$$

③ Find $\lim_{x\to\infty} \frac{e^x}{x}$.

$$\lim_{x\to\infty} \frac{e^x}{x} \quad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} \frac{e^x}{1} = \frac{\infty}{1} = \infty$$

④ Find $\lim_{x\to\infty} \frac{\ln(\ln x)}{\sqrt{x}}$.

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{\sqrt{x}} \quad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{\frac{1}{2}(x)^{-1/2}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x\ln x}}{\frac{1}{2\sqrt{x}}}$$

$$= \lim_{x\to\infty} \frac{2\sqrt{x}}{x\ln x}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{x}\ln x} = \frac{2}{\infty} = 0$$

Note:- For any other form we need to bring those indeterminate forms to either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ & then apply L'Hospital.

(*) Don't apply L'Hospital directly on any indeterminate form.

Form $0 \cdot \infty$:

Find $\lim_{x \to \infty} x e^{-x^2}$.

Note: direct substitution gives $\infty \cdot 0$

$$\lim_{x \to \infty} x \cdot e^{-x^2} \quad [\infty \cdot 0]$$

$$= \lim_{x \to \infty} x \cdot \frac{1}{e^{x^2}}$$

$$= \lim_{x \to \infty} \frac{x}{e^{x^2}} \quad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x \to \infty} \frac{1}{e^{x^2} \cdot 2x}$$

$$= \frac{1}{\infty \cdot \infty} = 0.$$

Form $\infty - \infty$:

Find $\lim_{x \to \infty} (x - \ln x)$.

$$\lim_{x \to \infty} (x - \ln x) \quad [\infty - \infty]$$

$$= \lim_{x \to \infty} x \left(1 - \frac{\ln x}{x}\right)$$

$$= \infty \cdot (1 - 0)$$

$$= \infty$$

Since $\lim_{x \to \infty} \frac{\ln x}{x} \left[\frac{\infty}{\infty}\right]$

$$= \lim_{x \to \infty} \frac{1/x}{1}$$

$$= \lim_{x \to \infty} \frac{1}{x} = 0$$

Form $1^\infty, \infty^0, 0^0$:

Note: The procedure is same for all these types.

Find $\lim_{x\to\infty} x^{1/x}$.

$$\lim_{x\to\infty} x^{1/x} \quad [\infty^0]$$

First we consider $y = x^{1/x}$, then take ln on both sides

ie, $\ln y = \ln(x^{1/x}) = \frac{1}{x}\ln x$ [by properties of ln]

Now we take limit on both side:

$$\lim_{x\to\infty}(\ln y) = \lim_{x\to\infty}\left(\frac{1}{x}\ln x\right)$$
$$= \lim_{x\to\infty}\left(\frac{\ln x}{x}\right) \quad \left[\frac{\infty}{\infty}\right]$$
$$= \lim_{x\to\infty}\left(\frac{1/x}{1}\right) = 0$$

$$\Rightarrow \lim_{x\to\infty}(\ln y) = 0$$

Since ln is continuous, then $\lim_{x\to a}$ & ln commutes.

ie, $\lim_{x\to a}(\ln f(x)) = \ln\left(\lim_{x\to a} f(x)\right)$

$$\Rightarrow \ln\left(\lim_{x\to\infty} y\right) = 0$$

$$\Rightarrow \lim_{x\to\infty} y = 1$$

ie, $\lim_{x\to\infty} x^{1/x} = 1$